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Poiseuille flow at arbitrary Knudsen numbers and tangential momentum accommodation

By P. E. SUETIN, B. T. PORODNOV,
V. G. CHERNJAK AND S. F. BORISOV
Urals Politechnical Institute, Sverdlovsk K-2, U.S.S.R.

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A Poiseuille-flow problem in a cylindrical capillary in the whole range of Knudsen numbers with incomplete tangential momentum accommodation of molecules incident on the wall has been worked out. The linear non-homogeneous integral equation for the macroscopic gas velocity flow has been solved by the Bubnov–Galerkin method. For a limited range of Knudsen numbers, generally known results have been obtained.

An experimental investigation of the rare gases helium, neon and argon in the range of Knudsen numbers 10^3–10^{-3} has been made on packets consisting of 10 and 100 glass capillaries with molten walls. Comparison of theoretical and experimental data enables us to define both slip constants and tangential momentum accommodation coefficients. In the free-molecule flow regime the accommodation coefficients are 0·935, 0·929 and 0·975 for helium, neon and argon, respectively. In the viscous slip-flow regime these coefficients are equal to 0·895, 0·865 and 0·919, respectively. This difference in the tangential momentum accommodation coefficients is, most probably, due to the variable density of adsorbed molecules coating the capillary wall. Gas viscosity coefficients which coincide with those of Kestin within 0·5 % have also been calculated. Argon was used as the calibrating gas.

1. Introduction

Many kinetic and equilibrium phenomena on the boundary between a gas and a solid body (adsorption, slip, temperature jump, etc.) are determined by the interaction between a gas molecule and a solid surface. Slip flow on the surface is a good example of such an interaction. It is characterized by the tangential momentum accommodation coefficient of gas molecules incident on the solid surface. In the Maxwellian interpretation this is manifested by some molecules being reflected specularly.

The comparison of experimental and theoretical data for a rarefied gas flow in several common geometries shows the following.

(i) On the one hand, the observed gas flow rates in free-molecule flow are higher than the theoretical ones (in case of completely diffuse scattering) if the capillary walls are highly polished or molten (Knudsen 1909; Davis, Levenson & Milleron 1964).

(ii) On the other hand, the observed gas flow rates are lower than those predicted theoretically if surface roughness is created artificially (Davis *et al.* 1964; Gade 1913; Huggel 1952; Lund & Berman 1966).

As has already been shown, by Davis *et al.* (1964), de Marcus (1959) and Porodnov *et al.* (1972), the difference between experimental and theoretical results can be explained by taking into consideration capillary surface macroroughness where diffuse scattering occurs. Unfortunately, calculations of flows near rough surfaces in intermediate and viscous gas flow regimes are rather complicated.

To make theoretical analysis and experimental observation less complicated it is better to use capillaries with highly polished or molten surfaces which can be considered smooth. In this case, as was mentioned above, the observed gas flow rates are higher than theoretical ones in the whole range of Knudsen numbers, this difference essentially depending on the kind of gas (Lund & Berman 1966; Borisov, Porodnov & Suetin 1972).

It is quite natural to suppose that, if gas transfer takes place in a capillary, the increase in the observed gas flow rate (compared with that predicted theoretically) is due to conservation of the tangential momentum component of molecules incident on a wall after their reflexion.

In other words, in order to find close agreement between experimental and theoretical results, it is necessary to consider the boundary conditions (proposed by Maxwell) where molecules are reflected diffusely or specularly. The introduction of such boundary conditions can be justified by another experiment, namely, by the interaction of molecular beams with solid surfaces (Hurlbut 1963).

Cercignani & Pagani (1969) have already proposed a number of simplifying assumptions in solving this problem. In particular, they have proposed that the distribution function perturbation on a surface is independent of the velocity of reflected molecules and, therefore, is constant. In our opinion, such an assumption is not sufficiently accurate.

In this paper a theoretical investigation of gas flow in a cylindrical capillary with smooth walls with tangential momentum component accommodation is proposed. A brief account of the experimental technique is also given. Accommodation coefficients of some gases are defined and experimental and theoretical results are compared.

2. Theory

2.1. *Definition of the problem and basic equations*

Let us consider a steady isothermal gas flow in a capillary due to a pressure gradient directed along the capillary axis. The capillary is supposed to be so long that the end-effects can be neglected; the gas and the walls are at the same temperature T. So, to find the net gas flow rate it is necessary to know the gas velocity at any point of the capillary cross-section, whose area is Σ; $\mathbf{r}$ is the radius vector to this point (see figure 1).

The velocity distribution function f must satisfy the Boltzmann equation, which is of the form

$$\mathbf{v}.\frac{\partial f}{\partial \mathbf{r}}+v_z\frac{\partial f}{\partial z}=\mathscr{L}(f), \tag{1}$$

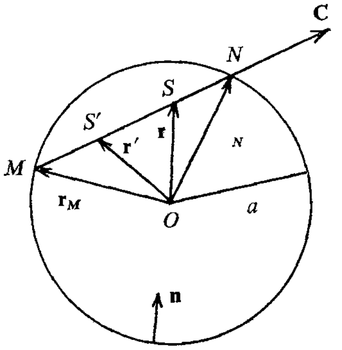

FIGURE 1. Problem geometry.

where z is a co-ordinate along the capillary axis, v_z is the z component of the absolute molecule velocity, $\mathbf{v}$ is the two-dimensional velocity vector in the capillary cross-section and $\mathscr{L}(f)$ is the intermolecular collision operator.

Further, we shall use the BGK model (Bhatnager, Gross & Krook 1954) for the operator $\mathscr{L}(f)$. Then, (1) may be written as

$$\mathbf{v}.\frac{\partial f}{\partial \mathbf{r}}+v_z\frac{\partial f}{\partial z} = \tau^{-1}(f_{eq}-f), \tag{2}$$

$$f_{eq} = n(z)\,(2\pi R_0 T)^{-\frac{3}{2}}\exp-[v^2+(v_z-u)^2]/2R_0T, \tag{2a}$$

where f_{eq} is the local Maxwellian distribution function, u is the macrosopic gas velocity, R_0 is the gas constant and τ^{-1} is the collision frequency, which, according to Halway (1967), is defined by

$$\tau^{-1} = p/\eta, \quad p = nkT, \quad \eta = 0{\cdot}5nm\lambda v_t. \tag{3}$$

In (3) we have employed the definition of the absolute pressure p and viscosity coefficient η resulting from the Chapman–Enskog approximation for hard spherical molecules (λ is the mean free path of molecules, v_t is the thermal velocity, n is the gas density and m is the mass of a molecule).

It is supposed that the gas perturbation is so small that the distribution function f only slightly differs from the Maxwellian one f_0, i.e. that f can be written as

$$f(z,\mathbf{r},C_z,\mathbf{C}) = f_0(z)\,[1+\phi(\mathbf{r},C_z,\mathbf{C})], \tag{4}$$

$$f_0(z) = n(z)\,(2\pi R_0T)^{-\frac{3}{2}}\exp[-C^2-C_z^2], \tag{4a}$$

where $\mathbf{C}$ and C_z are non-dimensional components of absolute molecule velocity:

$$(\mathbf{C},C_z) = (\mathbf{v},v_z)\,(2R_0T)^{-\frac{1}{2}}.$$

By introducing the linearized local Maxwellian function f_{eq} in (2) (considering (3)), then multiplying (2) by the capillary radius a and $(2R_0T)^{-\frac{1}{2}}$, we obtain a non-dimensional equation for the pertubation function ϕ:

$$\mathbf{C}.(\partial\phi/\partial\mathbf{R})+C_z\nu = \delta[2C_zU-\phi(\mathbf{R},C_z,\mathbf{C})]. \tag{5}$$

In (5) we have introduced the non-dimensional variables

$$\mathbf{R} = \frac{\mathbf{r}}{a}, \quad \nu = \frac{1}{n}\frac{dn}{dZ}, \quad Z = \frac{z}{a}, \quad \delta = \frac{\pi^{\frac{1}{2}}}{2}\frac{a}{\lambda} \equiv \frac{\pi^{\frac{1}{2}}}{2}\frac{1}{Kn};$$

here, Kn is the Knudsen number. The macroscopic non-dimensional gas velocity is

$$U(R) = \frac{1}{\pi^{\frac{3}{2}}}\iiint_{-\infty}^{\infty} \exp(-C^2-C_z^2).C_z\phi(\mathbf{R},C_z,\mathbf{C})\,dC_z\,d\mathbf{C}. \tag{6}$$

Following Cercignani & Sernagiotto (1966), we introduce the perturbation function

$$\Phi(\mathbf{R},\mathbf{C}) = \frac{1}{\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty} \exp(-C_z^2).C_z\phi(\mathbf{R},C_z,\mathbf{C})\,dC_z. \tag{7}$$

Then the basic kinetic equation (5) for the perturbation function $\Phi(\mathbf{R},\mathbf{C})$ takes the form

$$\mathbf{C}.[\partial\Phi(\mathbf{R},\mathbf{C})/\partial\mathbf{R}]+\tfrac{1}{2}\nu = \delta[U(R)-\Phi(\mathbf{R},\mathbf{C})]. \tag{8}$$

Equation (8) can be integrated along the arbitrary chosen direction $\mathbf{C}$ (Marchuk 1961; Narasimha 1961), see figure 1. After integration we obtain

$$\Phi(\mathbf{R},\mathbf{C}) = \Phi^{+}(\mathbf{R}_M,\mathbf{C})\exp(-\delta S/|\mathbf{C}|)+\int_0^S \frac{\delta U-\frac{1}{2}\nu}{|\mathbf{C}|}\exp[-\delta(S-S')/|\mathbf{C}|]\,dS', \tag{9}$$

$$S = |\mathbf{R}_M-\mathbf{R}|, \quad S-S' = |\mathbf{R}-\mathbf{R}'|.$$

In (9) $\mathbf{R}_M$ is the non-dimensional radius vector to the point M of the capillary surface, $|\mathbf{R}_M| = 1$; $\Phi^{+}(\mathbf{R}_M,\mathbf{C})$ is the perturbation function for the molecules reflected from the point M in the direction $\mathbf{C}$ (see figure 1).

The function $\Phi^{+}(\mathbf{R}_M,\mathbf{C})$ must be derived from the boundary conditions imposed on the distribution function $f(Z,\mathbf{r},C_z,\mathbf{C})$. Generally speaking, gas-molecule interaction with a hard wall leads to a basic difference between the velocity distributions of molecules incident on the wall and those reflected from it. The distribution function near the boundary is singular at the plane $\mathbf{n}.\mathbf{C} = 0$ in velocity space ($\mathbf{n}$ is the normal to the surface surrounding the gas). Therefore, while formulating macrosopic boundary conditions it is necessary to distinguish distribution functions of molecules incident on the surface and those reflected from it:

$$f(Z,\mathbf{R}_M,C_z,\mathbf{C}) = \begin{cases} f^{+}(Z,\mathbf{R}_M,C_z,\mathbf{C}) & (\mathbf{n}.\mathbf{C} > 0), \\ f^{-}(Z,\mathbf{R}_M,C_z,\mathbf{C}) & (\mathbf{n}.\mathbf{C} < 0). \end{cases} \tag{10}$$

Now we shall make use of a well-known Maxwellian model of the boundary conditions (Maxwell 1890). According to this model a certain fraction $1-\epsilon$ of the molecules is supposed to be reflected from the surface specularly, whereas the other molecules are scattered diffusely with a Maxwellian velocity distribution, i.e.

$$f^{+}(Z,\mathbf{R}_M,C_z,\mathbf{C}) = \epsilon f_0(Z)+(1-\epsilon)f^{-}(Z,\mathbf{R}_M,C_z,\mathbf{C}). \tag{11}$$

Besides, because of the axial symmetry of the problem the distribution function of molecules incident on the point M equals that of molecules incident on the point N (figure 1), i.e. we have

$$f^{-}(Z,\mathbf{R}_M,C_z,\mathbf{C}) = f^{-}(Z,\mathbf{R}_N,C_z,\mathbf{C}). \tag{12}$$

Now it is rather simple for us to obtain boundary and symmetry conditions of interest for the perturbation function $\Phi(\mathbf{R}, \mathbf{C})$ from (11) and (12) by considering (4) and (7):

$$\Phi^{+}(\mathbf{R}_M, \mathbf{C}) = (1-\epsilon)\,\Phi^{-}(\mathbf{R}_M, \mathbf{C}), \tag{13}$$

$$\Phi^{-}(\mathbf{R}_M, \mathbf{C}) = \Phi^{-}(\mathbf{R}_N, \mathbf{C}), \quad |\mathbf{R}_M| = |\mathbf{R}_N| = 1. \tag{14}$$

The perturbation function $\Phi^{-}(\mathbf{R}_N, \mathbf{C})$ of molecules incident of the point N may also be expressed in terms of $\Phi^{+}(\mathbf{R}_M, \mathbf{C})$ using (9) for if we suppose that $\mathbf{R} = \mathbf{R}_N$, then we have

$$\Phi^{-}(\mathbf{R}_N, \mathbf{C}) = \exp\left(-\frac{\delta l}{|\mathbf{C}|}\right)\left\{\Phi^{+}(\mathbf{R}_M, \mathbf{C}) + \int_0^l \frac{\delta U - \frac{1}{2}\nu}{|\mathbf{C}|}\exp\left(\frac{\delta S'}{|\mathbf{C}|}\right) dS'\right\}, \tag{15}$$

where $l = |\mathbf{R}_M - \mathbf{R}_N|$ is the length of the chord MN. Then the perturbation function $\Phi^{+}(\mathbf{R}_M, \mathbf{C})$ can be determined from (13)–(15) in the form

$$\Phi^{+}(\mathbf{R}_M, \mathbf{C}) = \frac{(1-\epsilon)\exp(-\delta l/|\mathbf{C}|)}{1-(1-\epsilon)\exp(-\delta l/|\mathbf{C}|)}\int_0^l \frac{\delta U - \frac{1}{2}\nu}{|\mathbf{C}|}\exp\left(\frac{\delta S'}{|\mathbf{C}|}\right) dS'. \tag{16}$$

Further, in the definition of the macroscopic gas velocity we use the perturbation function $\Phi(\mathbf{R}, \mathbf{C})$:

$$U(R) = \frac{1}{\pi}\iint_{-\infty}^{\infty} \exp(-C^2)\,\Phi(\mathbf{R}, \mathbf{C})\,d\mathbf{C} \quad (C \equiv |\mathbf{C}|).$$

Also, we introduce simultaneously a new function

$$\psi(R) = 1-(2\delta/\nu)\,U(R).$$

Then by considering (9) and (16) we get the integral equation

$$\psi(R) = 1+\frac{\delta}{\pi}\iint_{\Sigma} \psi(R')\,\frac{K(\mathbf{R}, \mathbf{R}')}{|\mathbf{R}-\mathbf{R}'|}\,d\mathbf{R}', \tag{17}$$

$$K(\mathbf{R}, \mathbf{R}') = \int_0^{\infty} \frac{1+(1-\epsilon)\exp[-(\delta/C)\,|\mathbf{R}_M-\mathbf{R}_N|]}{1-(1-\epsilon)\exp[-(\delta/C)\,|\mathbf{R}_M-\mathbf{R}_N|]}\exp\left(-C^2-\frac{\delta}{C}|\mathbf{R}-\mathbf{R}'|\right) dC.$$

The non-dimensional gas volume flow rate through the capillary cross-section Σ can be defined by

$$Q(\delta, \epsilon) = -\frac{2}{\nu\Sigma}\iint_{\Sigma} U(R)\,d\mathbf{R} = -\frac{1}{\delta}+\frac{2}{\delta}\int_0^1 \psi(R)\,R\,dR. \tag{18}$$

In the limiting cases of free-molecule flow ($\delta \to 0$) and viscous slip flow ($\delta \to \infty$), from (17) and (18) one can obtain the following results:

$$Q(\delta \to 0, \epsilon) = \frac{2-\epsilon}{\epsilon}\,\frac{8}{3\pi^{\frac{1}{2}}}, \tag{19}$$

$$Q(\delta \to \infty) = \frac{\delta}{4}+\frac{\pi^{\frac{1}{2}}}{2}\,\frac{2-\epsilon}{\epsilon}\,\sigma_T, \tag{20}$$

where the numerical value of the slip constant σ_T for the BGK model is 1·1466.

In the case of completely diffuse scattering of molecules by the wall, the results (19) and (20) are in full agreement with those of Cercignani & Sernagiotto (1966).

Thus, the problem is to solve the integral equation (17), for a capillary of arbitrary cross-section.

2.2. *Method of solution*

As (17) is a linear integral equation of the Fredholm type, of the second kind, to solve it we can employ the Bubnov–Galerkin method (Michlin 1970). For this purpose, we have chosen a system of basic functions $\{R^{2k}\}$ and, thus, we approximate the function $\psi(R)$ by the following series:

$$\psi(R) = \sum_{k=0}^{n} a_k(\delta, \epsilon)\, R^{2k}. \tag{21}$$

Quick convergence of the method used enables us to confine our attention to the first terms in the expansion (21) (the third term contributes to the sum not more than 1 %). Hence, we shall find the required function $\psi(R)$ as the sum

$$\psi(R) = a_0 + a_1 R^2. \tag{22}$$

After substituting (22) in (17) it is of great importance that the equation obtained should be orthogonal to each of the basic functions R^0 and R^2. As a result we obtain a simple system of algebraic equations defining the unknown coefficients a_0 and a_1:

$$\left.\begin{aligned} \alpha_{11} a_0 + \alpha_{12} a_1 &= \pi, \\ \alpha_{12} a_0 + \alpha_{22} a_1 &= \tfrac{1}{2}\pi. \end{aligned}\right\} \tag{23}$$

Here, we have introduced the following notation:

$$\left.\begin{aligned} \alpha_{11} &= \pi - \frac{\delta}{\pi} \iint_\Sigma d\mathbf{R} \iint_\Sigma K(\mathbf{R}, \mathbf{R}')\, \frac{d\mathbf{R}'}{|\mathbf{R}-\mathbf{R}'|}, \\ \alpha_{12} &= \frac{\pi}{2} - \frac{\delta}{\pi} \iint_\Sigma R^2\, d\mathbf{R} \iint_\Sigma K(\mathbf{R}, \mathbf{R}')\, \frac{d\mathbf{R}'}{|\mathbf{R}-\mathbf{R}'|}, \\ \alpha_{22} &= \frac{\pi}{3} - \frac{\delta}{\pi} \iint_\Sigma R^2\, d\mathbf{R} \iint_\Sigma K(\mathbf{R}, \mathbf{R}')\, \frac{R'^2\, d\mathbf{R}'}{|\mathbf{R}-\mathbf{R}'|}. \end{aligned}\right\} \tag{24}$$

As $\psi(R)$ is defined by (22) and (23) we can write (using (18)) the non-dimensional gas volume flow rate in the capillary as

$$Q(\delta, \epsilon) = -\frac{1}{\delta}\left[1 - \frac{\pi(\frac{1}{4}\alpha_{11} - \alpha_{12} + \alpha_{22})}{\alpha_{11}\alpha_{22} - \alpha_{12}^2}\right]. \tag{25}$$

To simplify (24) it is convenient to expand the function $K(\mathbf{R}, \mathbf{R}')$ in a series, since for any values of ϵ, δ, C and $\mathbf{R}$ the following condition must be satisfied:

$$(1-\epsilon)\exp[-(\delta/C)\,|\mathbf{R}_M - \mathbf{R}_N|] < 1.$$

In this case we have

$$\begin{aligned} K(\mathbf{R}, \mathbf{R}') = \sum_{j=0}^{\infty} (1-\epsilon)^j \{&\mathcal{J}_0(\delta|\mathbf{R}-\mathbf{R}'| + j\delta|\mathbf{R}_M - \mathbf{R}_N|) \\ &+ (1-\epsilon)\,\mathcal{J}_0(\delta|\mathbf{R}-\mathbf{R}'| + (j+1)\,\delta|\mathbf{R}_M - \mathbf{R}_N|)\}. \end{aligned} \tag{26}$$

Here $\mathcal{J}_0(x)$ is the Abramowitz (1953) transcendental function

$$\mathcal{J}_m(x) = \int_0^\infty t^m \exp(-t^2 - x/t)\, dt.$$

It should be mentioned that the series (26) converges very quickly, when $\epsilon \sim 1$. Therefore, we confine ourselves to two or three terms of this series for the purposes of comparison with experiment.

After not very complicated but very numerous transformations, the coefficients α_{11}, α_{12} and α_{22} (from expansion (26)) may be put in the form which is convenient for numerical calculations (see appendix).

We should like to point out that our results for completely diffuse scattering of molecules by a capillary wall ($\epsilon = 1$) are in close agreement with those of Cercignani & Sernagiotto (1966) in the whole range of Knudsen numbers (the divergence is less than 1 %).

Note that a formula for the non-dimensional gas volume flow rate has also been obtained by Cercignani & Pagani (1969). It is essentially different from (18)–(20) and for any value of δ it can be written as

$$Q(\delta, \epsilon) = Q(\delta, 1) + \pi^{\frac{1}{2}}(1-\epsilon)/\epsilon.$$

In addition, we have calculated the non-dimensional gas volume flow rate $Q(\delta, \epsilon)$ in the capillary for a tangential momentum accommodation coefficient ϵ in the range 0·8–1, at intervals of 0·02. We have confined ourselves to three terms in the series expansion of the coefficients γ_{ij} (see appendix).

Some of the numerical results are given in figure 2 below.

3. Experiments

3.1. *Measurement technique and treatment of experimental data*

Experimental investigations of the flow of inert gases such as helium, neon and argon were carried out by means of a method for unsteady flow proposed by Knudsen (1909). The essence of this method is to determine the small pressure-difference relaxation time as a function of the mean pressure in the capillary. This pressure difference has been observed at the ends of a capillary connecting two cavities.

In this paper we shall give only a brief description of the main features of the experiment since the basic features of the experimental installation and the measuring technique are described in detail in Borisov, Porodnov & Suetin (1972).

The experimental installation contains a working chamber consisting of two cavities joined by a packet of capillaries, a capacity differential micromanometer and a number of vacuum taps and absolute-pressure measuring devices. The working chamber, together with the micromanometer and taps, was placed in a thermostatically controlled atmosphere which was kept at room temperature during the experiment.

Experiments were performed on two packets consisting of 10 and 100 glass capillaries with molten walls with different mean radii but the same length-to-radius ratio $\sim$ 600. The deviation of the radii of the capillaries in a packet of 100 from the mean radius was not more than 2 %. To register a small pressure change in the working chamber a capacity differential micromanometer with numerical reading was used (Borisov, Kalinin, Porodnov & Suetin 1972). In the range of measured frequencies the micromanometer has a linear characteristic ~ 0·1 %, its sensitivity being $(3{\cdot}068 \pm 0{\cdot}004) \times 10^{-4}$ N/m² Hz.

From a conservation equation for the number of gas molecules in the cavities, whose volumes are V_1 and V_2, it is not difficult to find the pressure difference between the cavities as a function of time:

$$\Delta p = \Delta p_0 \exp(-ct), \quad c = \frac{M}{V[1+(b/V)p]}. \tag{27}$$

In (27), $V = V_1 V_2/(V_1+V_2)$ is the reduced system volume; Δp_0 and Δp are the pressure difference initially and at time $t > 0$ respectively; b is a constant depending on the flexure of the micromanometer membrane and was experimentally determined by two independent methods; M is the gas volume flow rate when the pressure difference between the capillary ends equals 1.

As the frequency of the registering device was proportional to the pressure difference between the cavities during the experiments, c can be calculated from this time-dependent frequency. Such calculations were performed with the help of a computer by the nonlinear least-squares method. In the calculation of c, the relative error in a viscous flow regime is ~ 0·1 %, while in a free-molecule flow regime it is ~ 1 %.

In (27) the value of b was determined from the dependence of c on p in viscous slip flow. In this flow regime the gas volume flow rate with a pressure difference equal to 1 is defined according to a well-known formula, which can be found in any book on kinetic theory:

$$M = N\frac{\pi a^4}{8\eta L}\left(1+4\frac{\xi}{a}\right)p \quad (\xi = \sigma\lambda). \tag{28}$$

In (28), N is a number of capillaries, a is the radius, L is the capillary length, η is the gas viscosity coefficient, ξ is the slip coefficient, σ is the slip constant, λ is the free mean path of a molecule and p is an absolute pressure. If the theoretical expression for the viscosity coefficient η in the Chapman–Enskog approximation ($\eta = 0{\cdot}5 nm\lambda v_t$) and the kinetic-theory definition of absolute pressure ($p = nkT$) are used, equation (28) takes the form

$$M = Ap+B. \tag{29}$$

Here, A is inversely proportional to the gas viscosity η and B is proportional to the slip constant σ.

From (27) and (29) it is easy to obtain the expression for c as a function of absolute pressure, depending on the parameters A, B and b.

A value of b equal to $0{\cdot}136 \pm 0{\cdot}002$ c.c./Torr was found by averaging all the b's which had been obtained while treating experimental results with the help of a computer for various gases, using the nonlinear least-squares method. This parameter was also found independently with the help of a cavity volume change measurement taken when a small flexure of the micromanometer membrane was observed.

The gas volume flow rate M was calculated from (27). Experimental data obtained with the help of packets consisting of 100 and 10 glass capillaries were the same within the limit of experimental error in the whole range of Knudsen numbers investigated.

In our experiments technically pure gases (purity being $> 99{\cdot}5\,\%$) were used. To eliminate the water vapour the gas was passed through a trap containing liquid nitrogen, and after that the experimental installation was filled with the gas. To investigate the influence of impurities on the gas volume flow rate we performed experiments with spectrally pure gases, helium and neon. No effect was found, however.

3.2. *Comparison of experimental and theoretical results*

Experimental data in the viscous slip-flow regime were treated with the help of the nonlinear least-squares method with three unknown parameters. From (28) and (29) we see that A describes the Poiseuille part of the flow, whereas B refers to that part of the flow which is characterized by a slip flow on the capillary surface. The third parameter, B, in (27) determines the correction due to volume changes caused by the micromanometer membrane flexure. The parameters A and B enable us to calculate either the capillary radius and the slip constant when the gas viscosity is known, or the gas viscosity and the slip constant when the capillary radius is known.

While calculating the mean capillary radius we have used gas viscosity coefficients from Kestin, Wakeham & Watanabe (1970) and Golubev & Gnezdilov (1971), whereas when calculating gas slip constants we have employed mean radii $\bar{a}_1 = (49{\cdot}86 \pm 0{\cdot}03) \times 10^{-4}\,\mathrm{cm}$ and $\bar{a}_2 = (19{\cdot}07 \pm 0{\cdot}03) \times 10^{-3}\,\mathrm{cm}$ for the packets consisting of 100 and 10 capillaries, respectively.

It should be mentioned that the measurement error resulting from using $\bar{a}_2$ is the same as the error obtained using capillary calibration by mercury.

The best way to compare experimental and theoretical data is to consider the non-dimensional gas volume flow rate Q as a function of the parameter δ [see equation (18)] over the whole range of Knudsen numbers investigated. The experimental value of Q was calculated from

$$Q = \frac{M}{\frac{1}{2}(2kT/m)^{\frac{1}{2}}\pi a^3/L}. \tag{30}$$

Experimental values of the non-dimensional gas volume flow rate Q as a function of δ for helium and argon as well as theoretical curves of $Q(\delta)$ calculated with the help of (25) are shown in figure 2. From figure 2 it may be clearly seen that there is an essential difference between the experimental results and the theoretical results calculated under the assumption that completely diffuse scattering of molecules takes place at the wall (curve 1). As was mentioned above, this difference can be explained by the fact that both specular and diffuse scattering in fact occur at the molten glass surface.

Therefore, we believe that the difference between the experimental and theoretical results is due to incomplete tangential momentum accommodation of molecules at the wall.

In the range $\delta < 3 \times 10^{-2}$ we have drawn the theoretical curves for the corresponding values ϵ, using the analytical expression for Q in the region near to

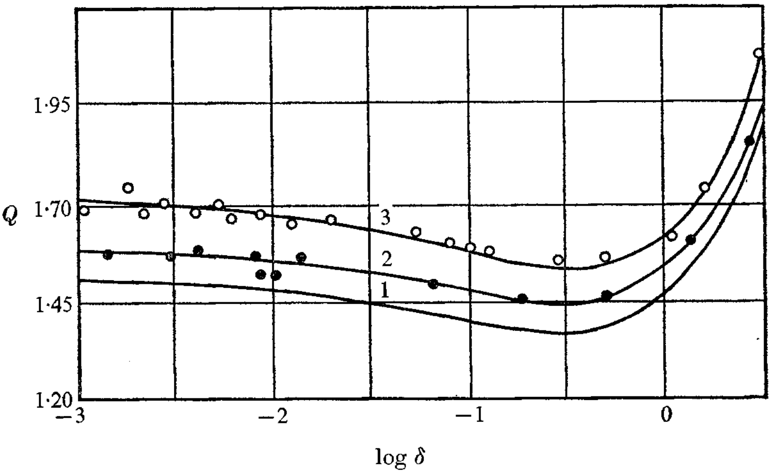

FIGURE 2. Non-dimensional gas volume flow rate dependence from δ and ϵ. ○, helium; ●, argon. Theoretical curves: 1, $\epsilon = 1$; 2, $\epsilon = 0{\cdot}975$; 3, $\epsilon = 0{\cdot}935$.

	Gas		
Coefficient	Helium	Neon	Argon
ω	1·139	1·152	1·051
$\pm\Delta\omega\times10^3$	7	6	10
ϵ	0·935	0·929	0·975

TABLE 1. The ratio ω and the accommodation coefficient ϵ in the free-molecule flow regime.

free-molecule flow. This allowed us to calculate the accommodation coefficients which are given in table 1, from the experimental data, the error in determining ϵ being less than 1 %. The value ω, which is the ratio of the experimental gas volume flow rate to the theoretical one, is also given in table 1.

Table 1 and figure 2 show that in the free-molecule flow regime the observed gas volume flow rates are 5–15 % higher than the theoretical ones. This divergence is much higher than the experimental error, which is ~ 1 % for light gases and ~ 1·5 % for heavy gases in the same flow regimes. Curves 2 and 3 (figure 2) were calculated by means of (25) for $\epsilon = 0{\cdot}975$ and $\epsilon = 0{\cdot}935$, respectively.

In the viscous slip-flow regime, slip constants σ were calculated according to (27) and (29). These constants and their mean-square errors are given in table 2. From table 2 it is seen that the experimental slip constants are higher than the theoretical ones for purely diffuse scattering of molecules by a wall [see equation (20)]. This difference enables us to calculate the tangential momentum accommodation coefficients ϵ in the viscous slip-flow regime, using (20). The calculated values of ϵ and their mean-square errors are also given in table 2.

While treating measurements in the viscous slip-flow regime it is important to choose experimental values of the gas viscosity. In the case of capillaries

	Gas		
Coefficient	Helium	Neon	Argon
σ	1·417	1·505	1·350
$\pm\Delta\sigma \times 10^3$	9	4	39
ϵ	0·895	0·865	0·919
$\eta \times 10^8$ (N s/m^2)	1972	3203	2272
$\pm\Delta\eta \times 10^8$ (N s/m^2)	3	5	7

TABLE 2. Experimental values of the slip constant σ, accommodation coefficient ϵ and viscosity η.

calibrated by mercury our method allows evaluation of the gas viscosity coefficients to within an accuracy of 0·2 %, when the Knudsen number is small enough and the slip contribution is of little significance. Thus, we obtained the gas viscosities given in table 2, which coincide with those of Kestin *et al.* (1970) to within 0·5 %. The calculated viscosity coefficients were adjusted to give values at 300 °K. Argon was used as a calibrating gas.

Appendix

The coefficients α_{11}, α_{12} and α_{22} were calculated with the help of the following formulae:

$$\alpha_{11} = \beta_{11} - \gamma_{11}, \quad \alpha_{12} = \beta_{12} - \gamma_{12}, \quad \alpha_{22} = \beta_{22} - \gamma_{22}, \quad \beta_{11} = 8I_1^0(\delta),$$

$$\beta_{12} = 8\{I_1^0(\delta) + \delta^{-2}[I_3^0(\delta) + 2I_2^1(\delta) - \tfrac{1}{8}\pi]\},$$

$$\beta_{22} = -\frac{\pi}{\delta^2} - \frac{10\pi^{\frac{1}{2}}}{\delta^3} + 8\left\{I_1^0(\delta) + \frac{6}{\delta^2}I_3^0(\delta) + \frac{10}{\delta^2}I_1^2(\delta) + \left(\frac{4}{\delta} + \frac{15}{\delta^3}\right)I_2^1(\delta)\right\},$$

$$\gamma_{11} = -8\sum_{j=1}^{\infty}(1-\epsilon)^j\{(j-1)I_1^0[(j-1)\delta] - 2jI_1^0(j\delta) + (j+1)I_1^0[(j+1)\delta]\},$$

$$\begin{aligned}\gamma_{12} = \gamma_{11} - \frac{8}{\delta}\sum_{j=1}^{\infty}(1-\epsilon)^j\Big\{&2(j+1)I_2^1[(j+1)\delta] - 2(j-1)I_2^1[(j-1)\delta]\\ &+\frac{2j-1}{\delta}I_3^0[(j-1)\delta] - \frac{4j}{\delta}I_3^0(j\delta) + \frac{2j+1}{\delta}I_3^0[(j+1)\delta]\Big\},\end{aligned}$$

$$\begin{aligned}\gamma_{22} = \gamma_{12} - \frac{8}{\delta}\sum_{j=1}^{\infty}(1-\epsilon)^j\Big\{&2(j+1)I_2^1[(j+1)\delta] - 2(j-1)I_2^1[(j-1)\delta]\\ &+\frac{6j-5}{\delta}I_3^0[(j-1)\delta] + \frac{4j}{\delta}I_3^0(j\delta) + \frac{6j+5}{\delta}I_3^0[(j+1)\delta]\\ &-\frac{3\pi^{\frac{1}{2}}}{\delta^2} - \frac{2(4j-5)}{\delta^2}I_4^1[(j-1)\delta] + \frac{4}{\delta^2}I_4^1(j\delta) + \frac{2(4j+5)}{\delta^2}I_4^1[(j+1)\delta]\\ &+\frac{4j}{\delta^3}I_5^0[(j-1)\delta] - \frac{8j}{\delta^3}I_5^0(j\delta) + \frac{4j}{\delta^3}I_5^0[(j+1)\delta]\Big\}.\end{aligned}$$

In the equations given above, the integrals of the form

$$I_m^n(\delta) = \int_0^1 v^n(1-v^2)^{\frac{1}{2}} \mathscr{J}_m(2\delta v)\, dv$$

satisfy the recurrence relation

$$2I_m^n(\delta) = (m-1)\, I_{m-2}^n(\delta) + 2\delta I_{m-3}^{n+1}(\delta).$$

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